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STIMULATED RAMAN SCATTERING INVOLVING TWO PLASMA WAVES

L. C. HIMMELL

UCLA • SCHOOL OF ENGINEERING AND APPLIED SCIENCE

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ABSTRACT

The parametric decay of an intense, coherent electromagnetic wave into a plasma wave and scattered electromagnetic waves in a homogeneous plasma (stimulated Raman scattering) is generalized to include the coupling of secondary waves in the plasma. The secondary waves consist of a plasma wave harmonic and its associated mixed electromagnetic-electrostatic sidebands. Explicit results are obtained for wave amplitudes and nonlinear growth rates when $\gamma t \gtrsim 1$, where γ is the linear growth rate of the fundamental. It is shown that, in this regime, the nonlinear growth rate γ_{NL} is $2\gamma/3$.

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I. INTRODUCTION

As is well known, there are many important fundamental processes associated with the interaction of an intense electromagnetic wave with a fully ionized plasma.[1,2,3] These processes have important implications for astrophysics as well as for laser studies. Extensive investigations have been made into stimulated Raman scattering, one important process in which an intense electromagnetic wave impinges on a plasma and scatters off an electron plasma mode.[4] If the incident wave is intense enough, it modifies the linear properties of the medium, thus creating many interesting new phenomena.[5] In this case, the Bohm-Gross mode, which normally has a frequency approximately equal to the plasma frequency, can exhibit a large amount of dispersion. In fact, if the incident field is large enough, we show that electrostatic modes exist with frequencies equal to half the plasma frequency, and that they couple strongly to a free plasma wave; i.e., a wave which satisfies the dispersion relation describing a system in which there is no incident field. In this case, although the back-scattered waves associated with the fundamental are several orders of magnitude larger than the fundamental itself, those associated with the harmonic are negligible. Since processes which involve coupling of energy into electrostatic waves with little back-scatter may be important for plasma heating, we study this process in detail. We find that this coupling results in a strong nonlinear modification of the usual linear growth rate associated with Raman scattering.

II. GENERAL FORMALISM

We assume that the unperturbed state consists of a large amplitude, plane polarized, coherent electromagnetic wave

$$\vec{E}_o = 2E_o \vec{\ell}_o \cos(\vec{k}_o \cdot \vec{x} - \omega_o t) \tag{1}$$

propagating in a homogeneous plasma, and that the pair $(\omega_o, \vec{k}_o)$ satisfies the usual dispersion relation

$$\omega_o^2 = \omega_p^2 + k_o^2 c^2 \tag{2}$$

in the absence of damping. This state is characterized by electrons oscillating rapidly in the field of the incident wave $\vec{E}_o$ with nonrelativistic quiver velocity $|\vec{V}_o| = e\vec{E}_o/m\omega_o$, while the ions form a stationary background*.

Next, we perturb the equilibrium and consider the mode coupling processes associated with the medium in the presence of the field $\vec{E}_o$. In particular, we consider scattering processes off electron modes and find that, in the regime $(V_o/c)^2 \sim \omega/\omega_o \ll 1$, modes with frequency of the order of half the plasma frequency exist and couple strongly to the Bohm-Gross mode.

We examine the effects of a density perturbation associated with an electrostatic wave $(\omega, \vec{k})$ and its first harmonic $(2\omega, 2\vec{k})$ and look for conditions under which the harmonic can grow significantly. Associated

*In our formalism, $\vec{V}_o$ corresponds to a complex velocity amplitude.

with these waves are their satellites $(\omega \pm j\omega_o, k \pm jk_o)$ and $(2\omega \pm j\omega_o, 2k \pm jk_o)$ where j is an integer. Waves associated with harmonics of the pump $(\omega_o, \vec{k}_o)$, however, give contributions of order V_o/c to the current densities and are therefore neglected. We assume that ω/k is much larger than the electron or ion thermal speeds and therefore we neglect any strong wave particle interactions. Under these conditions, we can use an essentially fluid model. All currents are first calculated from a zero temperature fluid model. Linear contributions to the currents are then modified to include thermal effects by introducting the appropriate linear dielectric constants and susceptibilities. The appropriate fluid equations are then simply

$$\begin{aligned} &\frac{\partial}{\partial t}\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{V} = -\frac{e}{m}(\vec{E} + \frac{1}{c}\vec{V}\times\vec{B}) \\ &\frac{\partial}{\partial t} n + \text{div}\, n\vec{V} = 0 \end{aligned} \tag{3}$$

where n and $\vec{V}$ refer to the density and velocity of the electrons. Ion contributions to the perturbed current densities are neglected since they are negligible in this frequency regime.

With this framework in mind, the fluid equations are solved for the Fourier amplitudes of n and $\vec{V}$ and the relevant current densities are computed. Next, time-varying equations for the wave packet amplitudes are found, assuming that all waves are coherent. In a Fourier representation, Eqs. (3) reduce to

$$\begin{aligned} &\vec{V}_k = -\frac{ie\vec{E}_k}{m\omega} - \frac{ie}{m\omega}\int \frac{\vec{k}_2}{\omega_2}(\vec{V}_{k_1}\cdot\vec{E}_{k_2})d\lambda + \frac{1}{\omega}\int (\vec{k}_2\cdot\vec{V}_{k_1})\left(\vec{V}_{k_2} + \frac{ie}{m\omega_2}\vec{E}_{k_2}\right)d\lambda \\ &n_k = \frac{k}{\omega}\cdot\int n_{k_1} V_{k_2} d\lambda \end{aligned} \tag{5}$$

where $d\lambda = dk_1 dk_2 \delta(k-k_1-k_2)$ and $dk_i = d\vec{k}_i d\omega$ $(i=1,2)$.[7] Labeling the sideband modes $(\omega\pm\omega_o, \vec{k}\pm\vec{k}_o)$ and $(2\omega\pm\omega_o, 2\vec{k}\pm\vec{k}_o)$ by $\pm, 2\pm$ respectively, and examining Eqs.(5) in a regime in which ponderomotive forces are important, we find that

$$n_k = \frac{\vec{k}\cdot\vec{V}_k}{\omega} n_o + (n_- - n_+) \frac{\vec{k}\cdot\vec{V}_o}{\omega}$$

$$\vec{V}_k = -\frac{ie}{m\omega}\vec{E}_k - \frac{ie}{m\omega}\left[\frac{\vec{k}_o}{\omega_o}(\vec{V}_+ + \vec{V}_-)\cdot\vec{E}_o - (\vec{k}_-\vec{E}_- + \vec{k}_+\vec{E}_+)\cdot\frac{\vec{V}_o}{\omega_o}\right]$$

$$-\frac{\vec{k}\cdot\vec{V}_o}{\omega}\left[(\vec{V}_+ + \frac{ie}{m\omega_o}\vec{E}_+) - (\vec{V}_- - \frac{ie}{m\omega_o}\vec{E}_-)\right] \tag{6a}$$

$$n_\pm = \pm\frac{\vec{k}_\pm}{\omega_o}\cdot[n_o\vec{V}_\pm \pm n_k\vec{V}_o]$$

$$\vec{V}_\pm = \mp\frac{ie}{m\omega_o}\vec{E}_\pm \mp \frac{ie}{m\omega_o}\left[\pm\frac{\vec{k}}{\omega}(\vec{V}_o\cdot\vec{E}_k) + \frac{\vec{k}_o}{\omega_o}(\vec{V}_k\cdot\vec{E}_o)\right]$$

$$+\frac{1}{\omega_o}\vec{k}\cdot\vec{V}_o(\vec{V}_k + \frac{ie}{m\omega}\vec{E}_k) \tag{6b}$$

$$n_{2k} = n_o\frac{\vec{k}\cdot\vec{V}_{2k}}{\omega} + (n_{2-} - n_{2+})\frac{\vec{k}\cdot\vec{V}_o}{\omega}$$

$$\vec{V}_{2k} = -\frac{ie}{2m\omega}\vec{E}_{2k} - \frac{ie}{2m\omega}\left[\frac{\vec{k}_o}{\omega_o}(\vec{V}_{2-} + \vec{V}_{2+})\cdot\vec{E}_o - (\vec{k}_{2-}\vec{E}_{2-} + \vec{k}_{2+}\vec{E}_{2+})\cdot\frac{\vec{V}_o}{\omega_o}\right]$$

$$-\frac{k\cdot V_o}{\omega}\left[(V_{2+} + \frac{ie}{m\omega_o}E_{2+}) - (V_{2-} - \frac{ie}{m\omega_o}E_{2-})\right] \tag{6c}$$

$$n_{2\pm} = \pm \frac{\vec{k}_{2\pm}}{\omega_o} \cdot [n_o \vec{V}_{2\pm} \pm n_{2k} \vec{V}_o]$$

$$\vec{V}_{2\pm} = \mp \frac{ie}{m\omega_o} \vec{E}_{2\pm} \mp \frac{ie}{m\omega_o} \left[\frac{\vec{k}_o}{\omega_o} (\vec{V}_{2k} \cdot \vec{E}_o) \pm \frac{\vec{k}}{\omega} (\vec{V}_o \cdot \vec{E}_{2k}) \right] +$$

$$+ \frac{2\vec{k} \cdot \vec{V}_o}{\omega_o} \left(\vec{V}_{2k} + \frac{ie}{2m\omega} \vec{E}_{2k} \right) \tag{6d}$$

where we have denoted the complex amplitudes $\vec{E}_{k_o}, \vec{V}_{k_o}$ by $\vec{E}_o, \vec{V}_o$ and neglected the effects of pump depletion. We have also assumed that $\vec{V}_{2k}$ may be of the same order as $\vec{V}_k$. These equations can be readily solved up to terms of order ω/ω_o . The results are as follows:

$$n_k = n_o \frac{\vec{k} \cdot \vec{V}_k}{\omega} \quad , \quad \vec{V}_k = - \frac{ie}{m\omega} \vec{P}_k \tag{7a}$$

$$n_\pm = \frac{\vec{k} \cdot \vec{V}_o}{\omega_o} n_k \quad , \quad \vec{V}_\pm = \mp \frac{ie}{m\omega_o} [\vec{E}_\pm \pm \frac{\vec{k}_\pm}{\omega} (\vec{V}_o \cdot \vec{P}_k)] \tag{7b}$$

$$n_{2k} = n_o \frac{\vec{k} \cdot \vec{V}_{2k}}{\omega} \quad , \quad \vec{V}_{2k} = - \frac{ie}{2m\omega} \vec{P}_{2k} \tag{7c}$$

$$n_{2\pm} = \frac{2\vec{k} \cdot \vec{V}_o}{\omega_o} n_{2k} \quad , \quad \vec{V}_{2\pm} = \mp \frac{ie}{m\omega_o} [\vec{E}_{2\pm} \pm \frac{\vec{k}_{2\pm}}{2\omega} (\vec{V}_o \cdot \vec{P}_{2k})] \tag{7d}$$

where

$$\vec{P}_k = \vec{E}_k - \frac{\vec{k}}{\omega_o} (\vec{E}_+ + \vec{E}_-) \cdot \vec{V}_o \tag{8}$$

with a corresponding expression for $\vec{P}_{2k}$. According to Eqs.(7), the ponderomotive forces modify the low frequency velocity amplitudes by changing $\vec{E}_k$ to $\vec{P}_k$ and $\vec{E}_{2k}$ to $\vec{P}_{2k}$, which in turn affects all density contributions. We note that, in this regime, $\vec{k}_\pm[(\vec{V}_o \cdot \vec{P}_k)/\omega]$ is of the same order as $\vec{E}_\pm$ and cannot be neglected. Therefore, there are

also ponderomotive modifications to the velocity amplitudes associated with the high frequency side bands. The second order contributions, obtained from Eqs.(5), are as follows:

$$n_k^{(2)} = \frac{\vec{k}}{\omega} \cdot \left[n_o \vec{V}_k^{(2)} + (n_-^{(2)} - n_+^{(2)}) \vec{V}_o \right] + N_k$$

$$\vec{V}_k^{(2)} = -\frac{ie}{m\omega} \frac{\vec{k}_o}{\omega_o} \left(\vec{V}_+^{(2)} + \vec{V}_-^{(2)} \right) \cdot \vec{E}_o + \frac{1}{\omega} \left(\vec{V}_-^{(2)} - \vec{V}_+^{(2)} \right) \vec{k} \cdot \vec{V}_o + \vec{V}_k' \tag{9a}$$

$$n_\pm^{(2)} = \frac{\vec{k}_\pm}{\omega_\pm} \cdot \left[n_o \vec{V}_\pm^{(2)} \pm n_k^{(2)} \vec{V}_o \right] + N_\pm$$

$$\vec{V}_\pm^{(2)} = -\frac{ie}{m\omega_\pm} \frac{\vec{k}_o}{\omega_o} \left(\vec{V}_k^{(2)} \cdot \vec{E}_o \right) \pm \frac{1}{\omega_\pm} (\vec{k} \cdot \vec{V}_o) \vec{V}_k^{(2)} + \vec{V}_\pm' \tag{9b}$$

$$n_{2k}^{(2)} = \frac{\vec{k}}{\omega} \cdot \left[n_o \vec{V}_{2k}^{(2)} + \left(n_{2-}^{(2)} - n_{2+}^{(2)} \right) \vec{V}_o \right] + N_{2k}$$

$$\vec{V}_{2k}^{(2)} = -\frac{ie}{2m\omega} \frac{\vec{k}_o}{\omega_o} \cdot \left(\vec{V}_{2+}^{(2)} + \vec{V}_{2-}^{(2)} \right) \cdot \vec{E}_o$$

$$+ \frac{1}{2\omega} \left(\vec{V}_{2-}^{(2)} - \vec{V}_{2+}^{(2)} \right) \vec{k} \cdot \vec{V}_o + \vec{V}_{2k}' \tag{9c}$$

$$n_{2\pm}^{(2)} = \frac{\vec{k}_{2\pm}}{\omega_{2\pm}} \cdot \left[n_o \vec{V}_{2\pm}^{(2)} \pm n_{2k}^{(2)} \vec{V}_o \right] + N_{2\pm}$$

$$\vec{V}_{2\pm}^{(2)} = -\frac{ie}{m\omega_{2\pm}} \frac{\vec{k}_o}{\omega_o} \left(\vec{V}_{2k}^{(2)} \cdot \vec{E}_o \right) + \frac{2\vec{k} \cdot \vec{V}_o}{\omega_o} \vec{V}_{2k}^{(2)} + \vec{V}_{2\pm}' \tag{9d}$$

where

$$N_k = \frac{\vec{k}}{\omega}\cdot\Big[n_{2k}\vec{V}_{-k} + n_{2+}\vec{V}_{-k-k_o} + n_{2-}\vec{V}_{-k+k_o} + n_{-k}\vec{V}_{2k} + n_{-k-k_o}\vec{V}_{2+} + n_{-k+k_o}\vec{V}_{2-}\Big]$$

$$\begin{aligned}
\vec{V}'_k = -\frac{ie}{m\omega}\Big[& \frac{\vec{k}}{\omega}\left(\vec{V}_{-k}\cdot\vec{E}_{2k}\right) + \frac{\vec{k}_{2+}}{\omega_{2+}}\left(\vec{V}_{-k-k_o}\cdot\vec{E}_{2+}\right) - \frac{\vec{k}_{2-}}{\omega_{2-}}\left(\vec{V}_{-k+k_o}\cdot\vec{E}_{2-}\right) \\
& + \frac{\vec{k}}{\omega}\left(\vec{V}_{2k}\cdot\vec{E}_{-k}\right) + \frac{\vec{k}_+}{\omega_+}\left(\vec{V}_{2+}\cdot\vec{E}_{-k-k_o}\right) + \frac{\vec{k}_-}{\omega_-}\left(\vec{V}_{2-}\cdot\vec{E}_{-k+k_o}\right)\Big] \\
& + \frac{1}{\omega}\Big[\left(2\vec{k}\cdot\vec{V}_{-k}\right)\vec{k}P_2 - \left(\vec{k}_{2+}\cdot\vec{V}_{-k-k_o}\right)\frac{\vec{k}_{2+}}{2}\,\vec{V}_o\cdot\vec{P}_{2k} \\
& - \left(\vec{k}_{2-}\cdot\vec{V}_{-k+k_o}\right)\frac{\vec{k}_{2-}}{2}\left(\vec{V}_o\cdot\vec{P}_{2k}\right) - \left(\vec{k}\cdot\vec{V}_{2k}\right)\vec{k}P^* \\
& - \left(\vec{k}_+\cdot\vec{V}_{2+}\right)\vec{k}_+\left(\vec{V}_o\cdot\vec{P}_k\right) - \left(\vec{k}_-\cdot\vec{V}_{2-}\right)\vec{k}_-\left(\vec{V}_o\cdot\vec{P}_k\right)\Big]\frac{ie}{m\omega\omega_o}
\end{aligned} \tag{10a}$$

$$N_\pm = \frac{\vec{k}_\pm}{\omega_\pm}\cdot\Big[n_{2k}\vec{V}_{-k\pm k_o} + n_{2\pm}\vec{V}_{-k} + n_{-k}\vec{V}_{2\pm}\Big]$$

$$\begin{aligned}
\vec{V}'_\pm = -\frac{ie}{m\omega_\pm}\Big[& \frac{\vec{k}}{\omega}\left(\vec{V}_{-k\pm k_o}\cdot\vec{E}_{2k}\right) + \frac{\vec{k}_{2\pm}}{\omega_{2\pm}}\left(\vec{V}_{-k}\cdot\vec{E}_{2\pm}\right) \\
& + \frac{\vec{k}}{\omega}\left(\vec{V}_{2\pm}\cdot\vec{E}_{-k}\right) + \frac{\vec{k}_\mp}{\omega_\mp}\left(\vec{V}_{2k}\cdot\vec{E}_{-k+k_o}\right)\Big] \\
& + \frac{1}{\omega_\pm}\Big[\left(2\vec{k}\cdot\vec{V}_{-k\pm k_o}\right)\vec{k}P_2 - \left(\vec{k}_{2\pm}\cdot\vec{V}_{-k}\right)\frac{\vec{k}_{2\pm}}{2}\left(\vec{V}_o\cdot\vec{P}_{2k}\right) \\
& - \left(\vec{k}\cdot\vec{V}_{2\pm}\right)\vec{k}P_2 + \left(\vec{k}_\mp\cdot\vec{V}_{2k}\right)\vec{k}_\pm\left(\vec{V}_o\cdot\vec{P}_{-k}\right)\Big]\frac{ie}{m\omega\omega_o}
\end{aligned} \tag{10b}$$

$$N_{2k} = \frac{\vec{k}}{\omega}\cdot\left[n_k\vec{V}_k + n_+\vec{V}_- + n_-\vec{V}_+\right]$$

$$\vec{V}'_{2k} = -\frac{ie}{2m\omega}\left[\frac{k}{\omega}(\vec{V}_k\cdot\vec{E}_k) + \frac{\vec{k}_+}{\omega_+}(\vec{V}_-\cdot\vec{E}_+) + \frac{\vec{k}_-}{\omega_-}(\vec{V}_+\cdot\vec{E}_-)\right]$$

$$+\frac{1}{2\omega}\left[(\vec{k}\cdot\vec{V}_k)\vec{k}P - (\vec{k}_-\cdot\vec{V}_+)\vec{k}_-(\vec{V}_o\cdot\vec{P}_k) - (\vec{k}_+\cdot\vec{V}_-)\vec{k}_+(\vec{V}_0\cdot\vec{P}_k)\right]\frac{ie}{m\omega\omega_o} \tag{10c}$$

$$N_{2\pm} = \frac{\vec{k}_{2\pm}}{\omega_{2\pm}}\cdot\left[n_k\vec{V}_\pm + n_\pm\vec{V}_k\right]$$

$$\vec{V}'_{2\pm} = -\frac{ie}{m\omega_{2\pm}}\left[\pm\frac{\vec{k}_\pm}{\omega_o}(\vec{V}_k\cdot\vec{E}_\pm) + \frac{k}{\omega}(\vec{V}_\pm\cdot\vec{E}_k)\right]$$

$$+\frac{1}{\omega_{2\pm}}\left[-(\vec{k}_\pm\cdot\vec{V}_k)\vec{k}_\pm(\vec{V}_o\cdot\vec{P}_k) + (\vec{k}\cdot\vec{V}_\pm)\vec{k}P\right]\frac{ie}{m\omega\omega_o} \tag{10d}$$

$$P = (\vec{E}_+ + \vec{E}_-)\cdot\vec{V}_o,\quad P_2 = (\vec{E}_{2+} + \vec{E}_{2-})\cdot\vec{V}_o \quad .$$

We now apply the theory to a Stokes mode, $D_- \ll D_+$, where[5]

$$D_\pm = k_\pm^{\,2}c^2 - \varepsilon_\pm\omega_\pm^{\,2} \tag{11}$$

with

$$\varepsilon_\pm = 1 - \omega_p^{\,2}/\omega_\pm^{\,2} \tag{12}$$

This implies that

$$E_+ \sim E_k \sim \frac{V_o}{c}E_- \tag{13a}$$

and

$$E_{2\pm} \sim \frac{V_o}{c}E_{2k} \quad . \tag{13b}$$

as we show below. With this ordering, the second order perturbations reduce to

$$n_k^{(2)} = 3n_o \frac{k^2}{\omega^2} (\vec{V}_{2k} \cdot \vec{V}_{-k}) \quad , \quad \vec{V}_k^{(2)} = \frac{\vec{k}}{\omega} (\vec{V}_{2k} \cdot \vec{V}_{-k}) \tag{14a}$$

$$n_\pm^{(2)} = \left(\frac{\vec{k} \cdot \vec{V}_o}{\omega_o}\right) n_k^{(2)} \quad , \quad \vec{V}_\pm^{(2)} = \left(\frac{\vec{k} \cdot \vec{V}_o}{\omega_o}\right) (\vec{V}_{2k} \cdot \vec{V}_{-k}) \frac{\vec{k}_\pm}{\omega} \tag{14b}$$

$$n_{2k}^{(2)} = \frac{3}{2} n_o \left(\frac{\vec{k} \cdot \vec{V}_k}{\omega}\right)^2 \quad , \quad \vec{V}_{2k}^{(2)} = \vec{k} \frac{V_k^2}{2\omega} \tag{14c}$$

$$n_{2\pm}^{(2)} = 2\left(\frac{\vec{k} \cdot \vec{V}_o}{\omega_o}\right) n_{2k}^{(2)} \quad , \quad \vec{V}_{2\pm}^{(2)} = \frac{V_k^2}{2\omega} \left(\frac{\vec{k} \cdot \vec{V}_o}{\omega_o}\right) \vec{k}_{2\pm} \quad . \tag{14d}$$

The current density perturbations, due to the response of the electrons to the electric and magnetic fields, are then as follows:

$$\vec{J}_k = \frac{i}{4\pi} \frac{\omega_p^2}{\omega} \vec{P}_k - \frac{e}{4\pi\omega} \cdot \frac{3}{2} \frac{\omega_p^2}{\omega^2} \frac{\vec{k}}{m} P_{2k} P_{-k} \tag{15a}$$

$$\vec{J}_\pm = \frac{i}{4\pi} \frac{\omega_p^2}{\omega_\pm} \left[\vec{E}_\pm \pm \vec{k}_\pm \left(\frac{\vec{V}_o \cdot \vec{P}_k}{\omega}\right) + \vec{V}_o \frac{\omega_o}{\omega} \frac{kP_k}{\omega}\right]$$

$$\mp \frac{e}{4\pi} \frac{3}{2} \frac{\omega_p^2}{\omega^2} \frac{\vec{V}_o}{m} \frac{k^2}{\omega^2} P_{2k} P_{-k} \tag{15b}$$

$$\vec{J}_{2k} = \frac{i}{4\pi} \frac{\omega_p^2}{2\omega} \vec{P}_{2k} + \frac{e}{4\pi\omega} \cdot \frac{3}{2} \vec{k} \frac{P_k^2}{m} \frac{\omega_p^2}{\omega^2} \tag{15c}$$

$$\vec{J}_{2\pm} = \frac{i}{4\pi} \frac{\omega_p^2}{\omega_{2\pm}} \left[\vec{E}_{2\pm} \pm \frac{\vec{k}_{2\pm}}{2\omega} \vec{V}_o \cdot \vec{P}_{2k} + \frac{k}{\omega} P_{2k} \frac{\omega_o}{2\omega} \vec{V}_o\right]$$

$$\pm \frac{e}{4\pi} \cdot \frac{3}{2} \frac{\omega_p^2}{\omega^2} \vec{V}_o \left(\frac{kP_k}{m\omega^2}\right)^2 \tag{15d}$$

Thermal corrections to the first order current density perturbations can be obtained by letting $\omega_p^2/\omega^2 \to -\chi(k,\omega)$, $\omega_p^2/\omega_\pm^2 \to -\chi(k_\pm,\omega_\pm)$, etc. The second term on the right side of Eqns (15b) and (15d) will be shown below to give a negligible contribution to Eqns (17a) and (17b), respectively. If we add thermal corrections to all other terms of first order, we obtain the following expressions for the first order current density perturbations:

$$\vec{J}_k^{(1)} = -\frac{i}{4\pi}\,\omega\chi\,(k,\omega)\,\vec{P}_k \tag{16a}$$

$$\vec{J}_\pm^{(1)} = -\frac{i}{4\pi}\,\omega_\pm\chi_\pm\vec{E}_\pm \mp \frac{i}{4\pi}\,\chi(k,\omega)\vec{V}_o kP_k \pm \frac{i\omega_p^2}{4\pi\omega_\pm}\,\vec{k}_\pm\left(\frac{\vec{V}_o\cdot\vec{P}_k}{\omega}\right) \tag{16b}$$

$$\vec{J}_{2k}^{(1)} = -\frac{i}{4\pi}\,2\omega\chi(2k,2\omega)\vec{P}_{2k} \tag{16c}$$

$$\vec{J}_{2\pm}^{(1)} = -\frac{i}{4\pi}\,\omega_{2\pm}\chi_{2\pm}\vec{E}_{2\pm} \mp \frac{i}{4\pi}\,\chi(2k,2\omega)\vec{V}_o 2kP_{2k} \pm \frac{i\omega_p^2}{4\pi\omega_{2\pm}}\,\vec{k}_{2\pm}\left(\frac{\vec{V}_o\cdot\vec{P}_{2k}}{2\omega}\right) \tag{16d}$$

Since the second order current density perturbations $\vec{J}_k^{(2)}$ and $\vec{J}_{2k}^{(2)}$ are both proportional to $\vec{k}$, the fields $\vec{E}_k$ and $\vec{E}_{2k}$ remain electrostatic in second order.

The Fourier-transformed wave equations for $\vec{E}_\pm$ and $\vec{E}_{2\pm}$ are as follows:

$$[(k_\pm^2-\omega_\pm^2/c^2)\underset{\sim}{I} - \vec{k}_\pm\vec{k}_\pm]\cdot\vec{E}_\pm = \frac{\omega_\pm^2}{c^2}\,\chi_\pm\vec{E}_\pm \pm \frac{\omega_\pm}{c^2}\,\chi\vec{V}_o kP_k \mp \frac{\omega_p^2}{c^2}\,\vec{k}_\pm\left(\frac{\vec{V}_o\cdot\vec{P}_k}{\omega}\right)$$
$$\mp \frac{3}{2}\, ie\,\frac{\omega_\pm}{c^2}\,\frac{\omega_p^2}{\omega^2}\,\frac{\vec{V}_0}{m}\,\frac{k^2}{\omega^2}\,P_{2k}P_{-k} \tag{17a}$$

$$[(k_{2\pm}^2 - \omega_{2\pm}^2/c^2)\underset{\sim}{I} - \vec{k}_{2\pm}\vec{k}_{2\pm}]\cdot\vec{E}_{2\pm} = \frac{\omega_{2\pm}^2}{c^2}\chi_{2\pm}\vec{E}_{2\pm} \pm \frac{\omega_{2\pm}}{c^2}\chi(2k,2\omega)\vec{V}_o 2kP_{2k}$$

$$\mp \frac{\omega_p^2}{c^2}\vec{k}_{2\pm}\left(\frac{\vec{V}_o\cdot\vec{P}_{2k}}{2\omega}\right)$$

$$\pm \frac{3}{2}\, ie\, \frac{\omega_{2\pm}}{c^2}\, \frac{\omega_p^2}{\omega^2}\, \frac{\vec{V}_o}{m}\, \frac{k^2}{\omega^2}\, P_k^2 \tag{17b}$$

where $\underset{\sim}{I}$ is the unit operator. These equations reduce to

$$[D_\pm \underset{\sim}{I} - k_\pm k_\pm c^2]\cdot\vec{E}_\pm = \vec{\vec{V}}_\pm P_k + \vec{\mathscr{J}}_\pm \tag{18}$$

and

$$[D_{2\pm}\underset{\sim}{I} - \vec{k}_{2\pm}\vec{k}_{2\pm}c^2]\cdot\vec{E}_{2\pm} = \vec{\vec{V}}_{2\pm}P_{2k} + \vec{\mathscr{J}}_{2\pm} \tag{19}$$

where

$$\vec{\vec{V}}_\pm = \vec{\vec{V}} \mp \omega_p^2\, \vec{k}_\pm\, \frac{\vec{k}\cdot\vec{V}_o}{k\omega} \tag{20a}$$

with

$$\vec{\vec{V}} = \omega_o \chi \vec{V}_o k \tag{20b}$$

and with a similar expression for $\vec{\vec{V}}_{2\pm}$. The normalized current densities, $\mathscr{J}_\pm$, and $\mathscr{J}_{2\pm}$, are given by

$$\vec{\mathscr{J}}_\pm = -\frac{3}{2}\, ie\omega_o\, \frac{\omega_p^2}{\omega^2}\, \frac{\vec{V}_o}{m}\, \frac{k^2}{\omega^2}\, P_{2k}P_{-k} \tag{21a}$$

and

$$\vec{\mathscr{J}}_{2\pm} = +\frac{3}{2}\, ie\omega_o\, \frac{\omega_p^2}{\omega^2}\, \frac{\vec{V}_o}{m}\, \frac{k^2}{\omega^2}\, P_k^2 \tag{21b}$$

where terms of order ω/ω_o are neglected. Similarly, E_k and E_{2k} satisfy the equations

$$\omega^2 E_k = - \omega^2 \chi(k,\omega) P_k + \mathcal{J}_k \tag{22a}$$

$$(2\omega)^2 E_{2k} = - (2\omega)^2 \chi(2k,2\omega) P_{2k} + \mathcal{J}_{2k} \tag{22b}$$

where

$$\vec{\mathcal{J}}_k = + \frac{3}{2} ie \frac{\omega_p^2}{\omega^2} \frac{\vec{k}}{m} P_{2k} P_{-k} \tag{23a}$$

and

$$\vec{\mathcal{J}}_{2k} = - 3ie \frac{\omega_p^2}{\omega^2} \frac{\vec{k}}{m} P_k^2 \quad . \tag{23b}$$

We now invert Eq.(18) and obtain

$$\vec{E}_\pm = \underset{\sim}{M}_\pm \cdot (\vec{\vec{V}}_\pm P_k + \vec{\mathcal{J}}_\pm) \tag{24a}$$

where

$$\underset{\sim}{M}_\pm = [\underset{\sim}{I} - \vec{k}_\pm \vec{k}_\pm c^2 / \omega_\pm^2 \varepsilon_\pm] / D_\pm \tag{24b}$$

Using Eqs.(22) and (24) to eliminate E_k, E_+ and E_- from Eq.(8), we obtain the following equation for P_k :

$$P_k \left[1 + \chi(k,\omega) + \frac{k\vec{V}_o}{\omega_o} \cdot \left((\underset{\sim}{M}_+ + \underset{\sim}{M}_-) \cdot \vec{V} - \omega_p^2 \left(\frac{\vec{k} \cdot \vec{V}_o}{k\omega} \right) (\underset{\sim}{M}_+ \vec{k}_+ - \underset{\sim}{M}_- \vec{k}_-) \right) \right]$$

$$= \frac{\mathcal{J}_k}{\omega^2} - \frac{k\vec{V}_o}{\omega_o} \cdot \underset{\sim}{M} \cdot \vec{\mathcal{J}} \tag{25a}$$

with

$$\underset{\sim}{M} \cdot \vec{\mathcal{J}} = \underset{\sim}{M}_+ \cdot \vec{\mathcal{J}}_+ + \underset{\sim}{M}_- \cdot \vec{\mathcal{J}}_- \tag{25b}$$

We note that

$$\frac{\vec{V}_o \cdot (\underset{\sim}{M}_+ + \underset{\sim}{M}_-) \cdot \vec{V}}{\omega_p^2 \frac{\vec{k}\cdot\vec{V}_o}{k\omega} \vec{V}_o \cdot (\underset{\sim}{M}_+ \cdot \vec{k}_+ - \underset{\sim}{M}_- \cdot \vec{k}_-)} \sim \omega_o/\omega \tag{26}$$

and therefore, that Eq.(25) reduces in lowest order to

$$P_k \Delta(\omega,k) = \frac{\mathscr{I}_k}{\omega^2} - \frac{k\vec{V}_o}{\omega_o} \cdot \underset{\sim}{M} \cdot \vec{\mathscr{I}} \tag{27}$$

where

$$\Delta(\omega,k) = 1 + \chi(k,\omega) + k^2\chi(k,\omega)\vec{V}_o \cdot (\underset{\sim}{M}_+ + \underset{\sim}{M}_-) \cdot \vec{V}_o \tag{28}$$

is the modified dielectric constant of the medium in the presence of $\vec{E}_o$.* Referring to Eqs.(21) and (23), we see that

$$\vec{\mathscr{I}}_\pm = -\mathscr{I}_k \frac{k\vec{V}_o}{\omega} \frac{\omega_o}{\omega} \tag{29}$$

which allows $\vec{\mathscr{I}}_\pm$ to be eliminated from Eq.(27) to give

$$P_k \Delta(\omega,k) = \frac{\mathscr{I}_k}{\omega^2}\left[1 + k^2\vec{V}_o \cdot (\underset{\sim}{M}_+ + \underset{\sim}{M}_-) \cdot \vec{V}_o\right]$$

$$= \frac{\mathscr{I}_k}{\omega^2\chi(k,\omega)} (-1 + \Delta(\omega,k)) \tag{30}$$

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*Eq.(28) can be readily shown to reduce to the result of Drake et al., if one remembers that $\vec{V}_o = -ie\vec{E}_o/m\omega_o$, while in Ref.5, $\vec{V}_o \equiv \frac{e\vec{E}_o}{m\omega_o}$.

where Eq.(28) has been used to further simplify the result. In a similar manner, we find that the equations for P_{2k} and $\Delta(2\omega,2k)$ are the same as Eqs.(30) and (28) respectively if (ω,k) is replaced by $(2\omega,2k)$. Using Eq.(30) to eliminate P_k from Eqs.(22) and (24, gives

$$\omega^2\Delta(\omega,k)E_k = \mathcal{J}_k \tag{31}$$

$$\omega^2\Delta(\omega,k)\vec{E}_\pm = -\omega_o k\,\mathcal{J}_k\,\underset{\sim}{M}_\pm\cdot\vec{V}_o \tag{32}$$

Similarly, we find

$$(2\omega)^2\Delta(2\omega,2k)E_{2k} = \mathcal{J}_{2k} \tag{33}$$

$$(2\omega)^2\Delta(2\omega,2k)\vec{E}_{2\pm} = -2\omega_o k\,\mathcal{J}_{2k}\,\underset{\sim}{M}_{2\pm}\cdot\vec{V}_o \tag{34}$$

For a Stokes mode, $D_- \sim \omega\omega_o$, $D_+ \sim \omega_o^2$ and $D_{2\pm} \sim \omega_o^2$ for $k \simeq 2k_o\cos\theta$.[5] Estimating the relative magnitudes of E_k and $E_\pm$ from Eqs.(31-32), assuming that $(V_o/c)^2 \sim \omega/\omega_o$, yields

$$\left|\frac{E_k}{E_+}\right| = \frac{1}{\omega_o k\left|\underset{\sim}{M}_+\cdot\vec{V}_o\right|} \sim \frac{D_+}{\omega_o kV_o} \sim \frac{\omega o}{kV_o} \sim \frac{1}{V_o/c} \tag{35a}$$

$$\left|\frac{E_k}{E_-}\right| = \frac{1}{\omega_o k\left|\underset{\sim}{M}_-\cdot\vec{V}_o\right|} \sim \frac{D_-}{\omega_o kV_o} \sim \frac{V_o}{c} \tag{35b}$$

Similarly, we find

$$\left|\frac{E_{2\pm}}{E_{2k}}\right| \sim \frac{\omega_o kV_o}{D_{2\pm}} \sim \frac{V_o}{c} \tag{36}$$

We now show that, for $\omega \simeq \frac{1}{2}\omega_\ell$, $\Delta(2\omega,2k) = 0$. According to Eq.(28),

$$\Delta(2\omega,2k) = \varepsilon(2\omega,2k) + (2k)^2\chi(2k,2\omega)\vec{V}_o\cdot(\underset{\sim}{M}_{2+} + \underset{\sim}{M}_{2}\,)\cdot\vec{V}_o \quad (37)$$

since $\varepsilon = 1 + \chi$. The last term on the right side of Eq.(37) is of order $(V_o/c)^2$, while $\varepsilon(\omega_\ell,2k) \simeq 0$. Therefore, $\Delta(2\omega,2k) = 0$ for some ω close to $\omega_\ell/2$.

If $\Delta(\omega_\ell/2,k) = 0$ for some $\vec{E}_o$, there may be significant coupling of energy between the waves $E_k,E_\pm$ and $E_{2k},E_{2\pm}$. The dispersion relation $\Delta(\omega,k) = 0$ for a Stokes mode in the regime $\omega >> kV_e$ reduces to[5]

$$(\omega-\Delta\omega)(\omega^2-\omega_\ell{}^2+i\omega^2\,\mathrm{Im}\chi) = -\,2\omega_o\omega_p{}^2\Psi^2(|V_o|^2/c^2) \quad (38)$$

where the geometry of the scattering is described by the following parameters:

$$\left.\begin{aligned}
\Psi &= |\sin\phi|\cos\theta \\
\sin^2\phi &= |\vec{k}_- \times \vec{V}_o|^2/k_-^2|V_o|^2 = |\vec{E}_o\cdot\vec{E}_-|^2/|E_oE_-|^2 \\
\Delta\omega &= c^2\,\frac{\vec{k}\cdot\vec{k}_o}{\omega_o} - \frac{c^2k^2}{2\omega_o} \\
D_-(k_-,\omega_-) &= 2\omega_o(\omega-\Delta\omega) \\
k &\simeq 2k_o\cos\theta
\end{aligned}\right\} \quad (39)$$

We neglect $\mathrm{Im}\chi(k,\omega)$ compared to $\mathrm{Im}\omega/\mathrm{Re}\omega$ and look for unstable modes with $\mathrm{Re}\,\omega \simeq \omega_\ell/2$. Let

$$F(\omega) = (\omega-\Delta\omega)(\omega^2-\omega_\ell{}^2) \quad (40)$$

We plot $F(\omega)$ vs. ω for real values of ω and determine $\Delta\omega$ such

that $F'(\omega_\ell/2) = 0$. This requires $\Delta\omega = -\frac{1}{4}\omega_\ell$. For this value of $\Delta\omega$, $F(\omega_\ell/2) = -\frac{9}{16}\omega_\ell^{\,3}$. Eq.(38) has three real roots for $A = 2\omega_o\omega_p^{\,2}\,\Psi^2(|V_o|^2/c^2) < \frac{9}{16}\omega_\ell^{\,3}$. For larger values of A , two real roots disappear, and the system becomes unstable, there being now one real root and two roots which are complex conjugates of each other. For back-scattering , $\Psi^2 \simeq 1$, which implies that

$$\left(\frac{|V_o|}{c}\right)^2 \gtrsim \frac{9}{32}\,\frac{\omega_\ell}{\omega_o} \tag{41}$$

is necessary for instability at this frequency. Setting

$$\left.\begin{aligned} \omega &= \omega_\ell/2 + i\gamma \\ A_o &= \frac{9}{16}\omega_\ell^{\,3} \\ \Delta\omega &= -\frac{1}{4}\omega_\ell + \Delta\tilde{\omega} \end{aligned}\right\} \tag{42}$$

and assuming that $\mathrm{Im}\gamma = 0$, the real and imaginary parts of Eq.(38) take the form

$$\left(\frac{7}{4}\omega_\ell - \Delta\tilde{\omega}\right)\gamma^2 - \frac{3}{4}\omega_\ell^{\,2}\Delta\tilde{\omega} = \Delta A \tag{43a}$$

$$\Delta\tilde{\omega}\,\omega_\ell + \gamma^2 = 0 \tag{43b}$$

where $\Delta A = A - A_o$ and the relevant solution for $\Delta\tilde{\omega}$,

$$\Delta\tilde{\omega} = \frac{5}{4}\omega_\ell\left[1 - \sqrt{1 + \Delta A/\frac{25}{16}\omega_\ell^{\,3}}\right] \tag{44}$$

can be expanded to give the following results for $\Delta\tilde{\omega}$ and γ^2

$$\Delta\tilde{\omega} = -\frac{2}{5}\frac{\Delta A}{\omega_\ell^2} \tag{45a}$$

$$\gamma^2 = \frac{2}{5}\frac{\Delta A}{\omega_\ell} \tag{45b}$$

In terms of the pump, we have

$$(\gamma/\omega_\ell)^2 = \frac{9}{20}\frac{\Delta E_o}{E_o}$$

where E_o is the threshold value corresponding to $\gamma = 0$ and ΔE_o is a small increment.

Applying standard techniques to Eqs.(31-34) enables us to obtain the following results for the equations describing the wave packet amplitudes, assuming that they are spatially independent:[7]

$$\left(\frac{\partial}{\partial t} - \gamma\right)\begin{bmatrix}\vec{E}_k \\ \vec{E}_\pm\end{bmatrix} = -\frac{i}{\left.\frac{\partial(\omega^2\Delta)}{\partial\omega}\right|_{\omega=\omega_\ell/2}} e^{-i\Delta\Omega_1 t}\mathcal{J}_k\begin{cases}\vec{k}/k \\ -\omega_o k\underset{\sim}{M}_\pm\cdot\vec{V}_o\end{cases} \tag{47a}$$

$$\left(\frac{\partial}{\partial t} + \gamma_L\right)\begin{bmatrix}\vec{E}_{2k} \\ \vec{E}_{2\pm}\end{bmatrix} = -\frac{i}{\left.\frac{\partial}{\partial\omega}(\omega^2\Delta)\right|_{\omega=\omega_\ell}} e^{-i\Delta\Omega_2 t}\mathcal{J}_{2k}\begin{cases}\vec{k}/k \\ -\omega_o 2k\underset{\sim}{M}_{2\pm}\cdot\vec{V}_o\end{cases} \tag{47b}$$

where

$$\Delta\Omega_1 = \omega(2k) + \omega(-k) - \omega(k) \tag{48a}$$

$$\Delta\Omega_2 = 2\omega(k) - \omega(2k) \tag{48b}$$

$$\left.\frac{\partial}{\partial\omega}(\omega^2\Delta)\right|_{\omega=\omega_\ell/2} = \frac{4}{3}\left(\frac{\gamma}{\omega_\ell}\right)^2\omega_\ell = \frac{9}{15}\frac{\Delta E_o}{E_o}\omega_\ell \tag{48c}$$

$$\frac{\partial}{\partial\omega}(\omega^2\Delta)\Big|_{\omega=\omega_\ell} = 2\omega_\ell \tag{48d}$$

Since $\Delta(2\omega,2k) \simeq \varepsilon(2\omega,2k)$, γ_L is just the usual linear Landau damping rate of a Bohm-Gross mode and, in this case, its sidebands.

If we neglect the slight frequency mismatches, $\Delta\Omega_1$ and $\Delta\Omega_2$, the equations for the wave amplitudes reduce to

$$\left(\frac{\partial}{\partial t} - \gamma\right)E_k(t) = c_1 P_{2k} P_{-k} \tag{49a}$$

$$\left(\frac{\partial}{\partial t} - \gamma\right)\vec{E}_\pm(t) = - c_1 \omega_o k \; \underset{\sim}{M}_\pm \cdot \vec{V}_o \; P_{2k} P_{-k} \tag{49b}$$

$$\left(\frac{\partial}{\partial t} + \gamma_L\right)E_{2k}(t) = - c_2 \; P_k^{\;2} \tag{49c}$$

$$\left(\frac{\partial}{\partial t} + \gamma_L\right)\vec{E}_{2\pm}(t) = c_2 \omega_o 2k \; \underset{\sim}{M}_{2\pm} \cdot \vec{V}_o \; P_k^{\;2} \tag{49d}$$

where

$$c_1 = \frac{9}{2}\,\frac{ek}{m\omega_\ell}\left(\frac{\gamma}{\omega_\ell}\right)^{-2} \tag{50a}$$

$$c_2 = 6\,\frac{ek}{m\omega_\ell} \quad . \tag{50b}$$

III. ANALYSIS

We now discuss the properties of Eqs.(49). We assume that at $t=0$, $E_{2k} \sim E_k$. Then, according to Eqs.(35-36), $E_+ \sim (V_o/c)E_k$, $E_- \sim E_k/(V_o/c)$, and $E_{2\pm} \sim (V_o/c)E_k$. Since $\omega >> kV_e$, $\gamma >> \gamma_L$. We show below that E_{2k} and $E_{2\pm}$ vary on a time scale $\sim 1/\gamma$. We therefore neglect γ_L in Eqs.(49). Also, a knowledge of $E_k(t)$, $E_{2k}(t)$ and the initial conditions of all the waves is sufficient to determine the nonlinear evaluation of $E_\pm$ and $E_{2\pm}$. One merely expresses Eqs.(49) in integral form as follows:

$$E_k(t) = E_k(0)e^{\gamma t} + \int_0^t d\tau\ e^{\gamma(t-\tau)} c_1 P_{2k}(\tau)P_{-k}(\tau) \tag{51a}$$

$$\vec{E}_\pm(t) = \vec{E}_\pm(0)e^{\gamma t} - \omega_o k(\underset{\sim}{M}_\pm \cdot \vec{V}_o) \int_0^t d\tau\ e^{\gamma(t-\tau)} c_1 P_{2k}(\tau)P_{-k}(\tau) \tag{51b}$$

with similar expressions for E_{2k} and $E_{2\pm}$. Solving for the integral in Eq.(51a) allows $E_\pm(t)$ to take the form

$$\vec{E}_\pm(t) = \vec{E}_\pm(0)e^{\gamma t} - \omega_o k\left(\underset{\sim}{M}_\pm \cdot \vec{V}_o\right)\left(E_k(t) - E_k(0)e^{\gamma t}\right) \tag{52}$$

The equation for $\vec{E}_{2\pm}(t)$ is obviously the same with $\gamma=0$ and (ω,k) replaced by $(2\omega,2k)$. Instead of analyzing $E_k(t)$ and its harmonic, it is more advantageous to examine $P_k(t)$ and $P_{2k}(t)$. These are the Fourier amplitudes of the total first order field that govern the motion of the electrons. In terms of P_k and P_{2k}, we find that

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \gamma\right)P_k &= c_1 P_{2k}P_{-k}(1 + k^2\vec{V}_o \cdot (\underset{\sim}{M}_+ \underset{\sim}{M}_-)\cdot\vec{V}_o) \\ &= c_1 P_{2k}P_{-k}(-1 + \Delta(\omega,k)\Big|_{\omega \approx \omega_\ell/2} \\ &= -c_1 P_{2k}P_{-k}\end{aligned} \tag{53}$$

and, similarly,

$$\frac{\partial}{\partial t} P_{2k} = c_2 P_k^{\ 2} \tag{54}$$

where we have taken appropriate linear combinations of Eqs.(49) and used Eqs.(28) and (37), remembering that $\Delta(\omega_\ell/2,k) \simeq \Delta(\omega_\ell,2k) \simeq 0$. We have therefore reduced the analysis of a six-wave system to an analogous one involving only two waves.

In order to analyze Eqs.(53-54), we first transform them to a dimensionless form and then express them in a polar representation. In terms of the nondimensional quantities $\tau = \gamma t$, $A_k = (\sqrt{c_1 c_2}/\gamma)P_k$, $A_{2k} = (c_1/\gamma)P_{2k}$, Eqs.(53-54) take the form

$$\left(\frac{\partial}{\partial\tau} - 1\right)A_k = - A_{2k}A_k^{\ *} \tag{55a}$$

$$\frac{\partial}{\partial\tau} A_{2k} = A_k^{\ 2} \tag{55b}$$

where, denoting the complex conjugate by $*$, $A_k^{\ *} = A_{-k}$, since $P_k^{\ *} = P_{-k}$ is a reality condition on the total electric field. Setting $A_k = a_k e^{i\phi_k}$ and $A_{2k} = a_{2k} e^{i\phi_{2k}}$, Eqs.(55) reduce to

$$\frac{d}{d\tau} a_k - a_k = - a_{2k} a_k \cos\phi \tag{56a}$$

$$\frac{d}{d\tau} a_{2k} = a_k^{\ 2} \cos\phi \tag{56b}$$

$$\frac{d}{d\tau} (a_k^{\ 2} a_{2k} \sin\phi) = 2a_k^{\ 2} a_{2k} \sin\phi \tag{56c}$$

where $\phi = \phi_{2k} - 2\phi_k$. Eq.(56c) is obtained from a linear combination of the imaginary parts of Eqs.(55a) and (55b). Its solution

$$a_k^2 a_{2k} \sin \phi = a_k^2(0) a_{2k}(0) \sin \phi(0) e^{2\tau} \tag{57}$$

allows Eqs.(56a) and (56b) to be further reduced to the form

$$\frac{dx}{d\tau} = \mp 2\sqrt{x^2y-1} \quad a^{1/3} \tag{58a}$$

$$\frac{dy}{d\tau} = \pm 2\sqrt{x^2y-1} \quad e^{2\tau} \, a^{1/3} \tag{58b}$$

where

$$x(\tau) = a_k^2(\tau) e^{-2\tau} \, a^{-2/3} \tag{59a}$$

$$y(\tau) = a_{2k}^2 (\tau) \, a^{-2/3} \tag{59b}$$

$$a = a_k^2(0) a_{2k}(0) \sin \phi(0) \tag{59c}$$

by allowing $\cos \phi$ to be eliminated. We note that

$$\frac{dx}{dy} = - e^{-2\tau} \tag{60}$$

a relation which enables Eqs.(58) to be readily analyzed in the phase plane. An upper bound may be obtained for $x(\tau)$ by integrating Eq.(60). We find that

$$\int_{x(0)}^{x(\tau)} dx = \int_{\tau'=0}^{\tau} [-d(e^{-2\tau'} y(\tau')) - 2e^{-2\tau'} y(\tau') d\tau'] \tag{61}$$

which reduces to

$$x(\tau) \leq y(0) + x(0) \tag{62}$$

where the inequality is a result of neglecting negative definite terms from the right side of Eq.(61). We show below, in fact, that $x(\tau)$ goes asymptotically to zero.

It is useful to examine the motion of x and y in the phase plane. According to Eqs.(58), the motion is bounded by the curve $x^2y-1 = 0$, and from Eq.(60), $|(dx/dy)|$ is equal to one at $\tau=0$, and decreases monotonically to zero. The motion can be analyzed further by examining an equation which describes the evolution of ϕ. We take an appropriate linear combination of the imaginary parts of Eqs.(55) and get

$$y^{1/2} \frac{d\phi}{d\tau} = (-\,xe^{2\tau}+2y)a^{1/3}\sin\phi \quad . \tag{63}$$

Turning points in Fig. I occur when $\phi = \pi/2$ or $3\pi/2$. Since $|dx| \leq |dy|$, between turning points, the trajectory to the right will always lie above the last trajectory to the left, and also above the trajectory to the left that immediately follows. The motion is relatively rapid, depending on the magnitude of $a^{1/3}$ (nominally in the range 1-10), except near the boundaries. As a first approximation, we assume straight-line motion between turning points except near the boundaries (see Fig. I).

We define two sets $\{x_j^{\ell}\}$ and $\{x_j^{r}\}$ such that x_j^{ℓ} labels the value of x at a left turning point and x_j^{r}, one that occurs to the right, where it is understood that j increases with time. It is clear from the analysis above that both sets are monotonically decreasing with lower bound equal to zero. Therefore, $x(\tau)$ goes to zero asymptotically.

Nonlinear growth rates may be determined from an analysis of Eq.(63). First, we note that the signed square root in Eqs.(58) is proportional to $\cot\phi$. According to Eq.(57), if $\phi(0) \neq 0$ or π , then $\phi(\tau)$ never goes to 0 or π in finite time, unless perhaps x or y becomes infinite at that time. The motion is therefore restricted to values in the half circle, $0 < \phi < \pi$ or $\pi < \phi < 2\pi$. We assume for definiteness that $0 < \phi(0) < \pi$, and note that $\phi'(\tau)$ is equal to zero somewhere an every trajectory between turning points. Let $\{x_j = x(\tau_j)\}$ be the set of values of x such that $\phi'(\tau_j) = 0$. According to Eq.(63), at this value of $\tau = \tau_j$

$$- x(\tau_j)e^{2\tau_j} + 2y(\tau_j) = 0 \quad . \tag{64}$$

Not only does $x(\tau_j)$ go to zero as $j \to \infty$, but also $dx/d\tau$ goes to zero with increasing j as exhibited in Fig. I. This implies that

$$x^2(\tau_j)y(\tau_j) - 1 \to 0 \tag{65}$$

asymptotically, as we see from Eqs.(58). The envelopes of the curves describing $x(\tau)$ and $y(\tau)$ are now readily determined by examining their values at the turning points. After several trajectories between turning points, the motion will be nearly horizontal, so that slight changes in the slope of the line $-xe^{2\tau}+2y = 0$ will intersect the curve $x^2y=1$ at neighboring turning points. Therefore,

$$x^2y=1 \ , \qquad x = 2e^{-2\tau}(1+\delta)y \ , \qquad 0 < \delta << 1 \tag{66a}$$

(at a left turning point)

while

$$x^2y=1 \ , \qquad x = 2e^{-2\tau}(1-\delta')y \ , \qquad 0 < \delta' << 1 \tag{66b}$$

(at a right turning point)

from which we find

$$\underset{\tau \geq \tau_j^{\ell}}{\text{Max}}\ x = x_j^{\ell} = 2^{1/3}(1+\delta)^{1/3} e^{-2/3\,\tau_j^{\ell}} \tag{67a}$$

$$\underset{\tau \leq \tau_j^{r}}{\text{Max}}\ y = y_j^{r} = \frac{e^{4/3\,\tau_j^{r}}}{[4(1-\delta')]^{1/3}} \tag{67b}$$

or in terms of $|P_k|$ and $|P_{2k}|$,

$$\underset{\tau \leq \tau_j^{\ell}}{\text{Max}}\ |P_k| = \frac{\gamma}{\sqrt{c_1 c_2}}\ a^{1/3}[2(1+\delta)]^{1/6}\ e^{2/3\,\tau_j^{\ell}} \tag{68a}$$

$$\underset{\tau \leq \tau_j^{r}}{\text{Max}}\ |P_{2k}| = \frac{\gamma}{c_1}\ \frac{a^{1/3} e^{2/3\,\tau_j^{r}}}{[4(1-\delta')^2]^{1/6}} \tag{68b}$$

$$\left(\frac{\text{Max}\ |P_{2k}|}{\text{Max}\ |P_k|}\right)_{\tau<\tau_j^{r}} = \sqrt{\frac{4}{3}}\ \frac{1}{2^{2/3}}\ \frac{\gamma}{\omega_\ell}\ e^{2/3(\tau_j^{r} - \tau_j^{\ell})}\ , \qquad \tau_j^{r} > \tau_j^{\ell} \tag{68c}$$

Since $\tau \equiv \gamma t$, this implies that P_k and P_{2k} evolve nonlinearly with a growth rate $\gamma_{NL} = \frac{2}{3}\gamma$.

The case where $\phi(0) = 0$ or π must be considered separately. According to Eq.(57), this implies that $\phi(\tau) = \phi(0)$ for all τ . In that case, Eqs.(56a) and (56b) reduce to

$$\frac{d}{d\tau}\ a_k^2 = 2a_k^2(1 \mp a_{2k}) \tag{69a}$$

$$\frac{d}{d\tau}\ a_{2k} = \pm\ a_k^2 \tag{69b}$$

$$\phi(\tau) = \begin{cases} 0 \\ \pi \end{cases} \tag{69c}$$

from which another constant of the motion may be deduced, namely,

$$a_k^2(\tau) + a_{2k}^2(\tau) \pm 2a_{2k}(\tau) = \lambda_\pm \tag{70}$$

A typical trajectory (a_k^2 vs. a_{2k}) is shown in Fig. 4. The orbit labeled (+) denotes $\phi(\tau) = 0$, while the orbit denoted by (-) refers to $\phi(\tau) = \pi$. The small circle corresponds to the initial point, $\tau = 0$. According to the figure, both orbits touch the a_{2k} axis and stop at a time which can be shown to correspond to a few units of τ. In this case, therefore, $P_k \to 0$, and P_{2k} saturates after a few linear e-folding times ($\gamma t \gtrsim 1$). (Negative values of a_{2k} on the orbit labeled (-) are due to the fact that the trajectory of A_k in the complex plane is along a ray, starting at a point in Quadrant I, passing through the origin, and ending in the third quadrant.)

IV. CONCLUDING REMARKS

We have examined the nonlinear properties associated with stimulated Raman wave-wave scattering off a natural plasma mode and a strongly modified plasma mode, and found that this complicated process, involving the mode-coupling of seven waves, can be reduced to an analysis of a two-wave system. An interesting feature of this system is that the waves have no asymptotic dependence on initial conditions, other than an assumption that E_k and its harmonic are initially of the same order. We can then determine analytically the asymptotic properties of the waves, and find that there is a strong nonlinear modification of the usual linear growth rate associated with stimulated Raman scattering and also that, although the harmonic has a rather small linear

damping rate γ_L , it grows nonlinearly at the same rate as its fundamental in the asymptotic limit.

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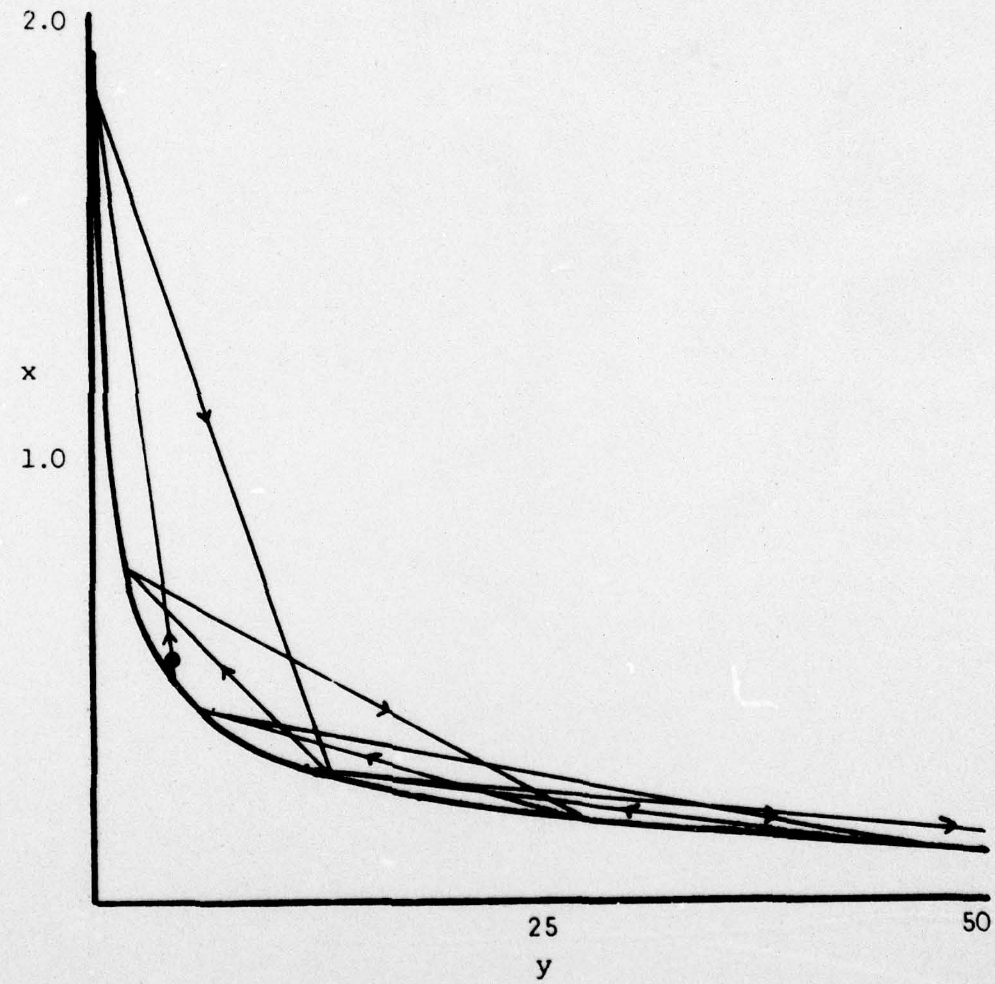

Fig. 1

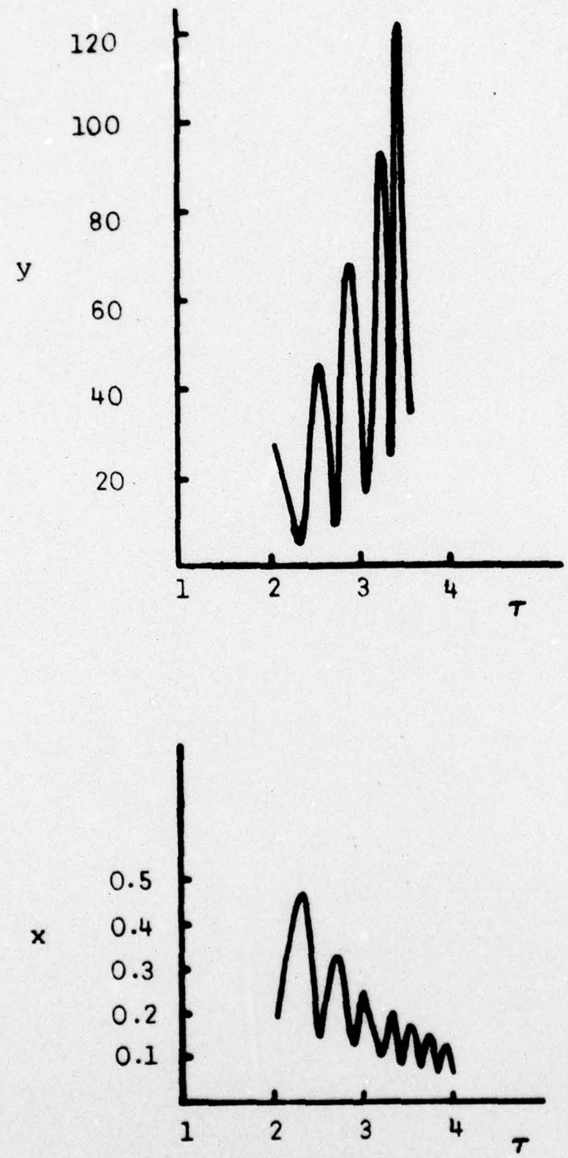

Fig. 2

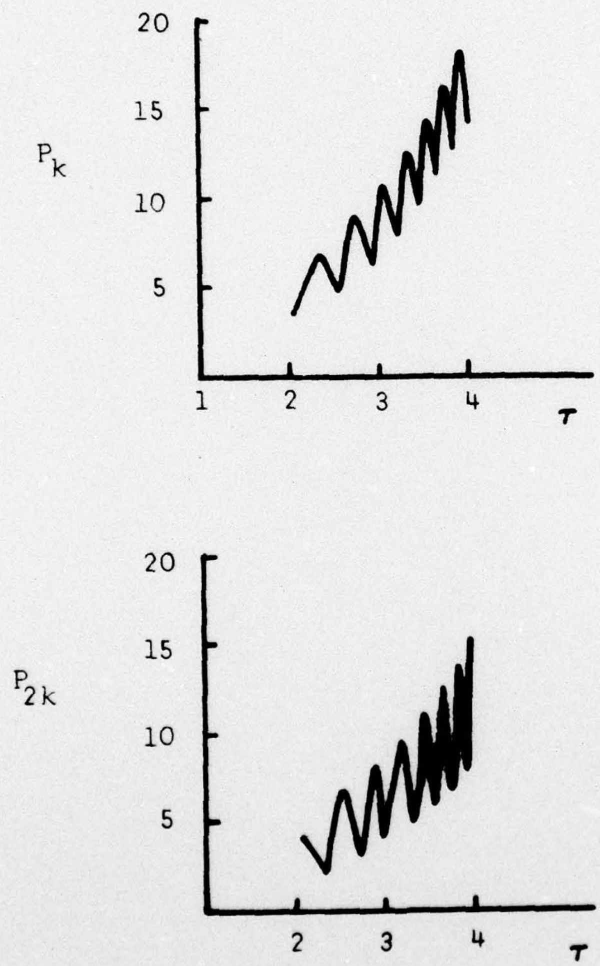

Fig. 3

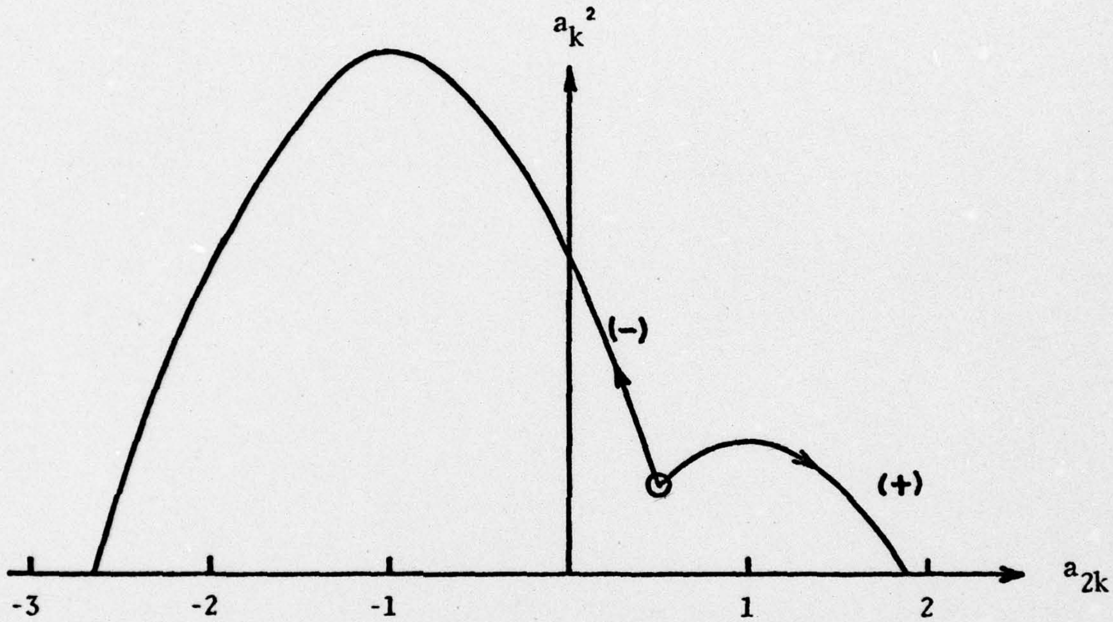

Fig. 4